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Point Deletions of Outerplanar Blocks

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Let G be a graph. If v is a vertex of G then the $(-1, v)$-subgraphs of G are defined to be the point deletions of G, except for $G \sim \{v\}$, with v labeled on each. This paper first classifies all outerplanar blocks which have a pair of v-isomorphic $(-1, v)$-subgraphs and next classifies all outerplanar blocks which have a pair of isomorphic point deletions. Finally, this classification is used to prove the Harary conjecture for outerplanar blocks, namely that an outerplanar block can be reconstructed from the isomorphism classes of its point deletions.

1. INTRODUCTION

Ulam [4] has conjectured that a graph G with at least three vertices is determined up to isomorphism by its collection of point deletions. In [3] Manvel credits Harary with the further conjecture that G is actually determined by the isomorphism classes of its point deletions. [3] also contains many references to the literature on this topic.

In [1] we established the Ulam conjecture for outerplanar graphs by working with partially labeled graphs. There, however, we required information on which outerplanar blocks can have a repeated v-labeled point deletion and our present Theorem 1 settles this question.

We next introduce some new families of outerplanar blocks and in Theorem 2 prove that, except for the obvious ones, these are the only outerplanar blocks which have a repeated point deletion.

Finally, as an application of Theorem 2, we show that the collection of all point deletions of G can be deduced from the isomorphism classes and so reduce the Harary conjecture for outerplanar blocks to the Ulam conjecture for outerplanar blocks.

2. PRELIMINARIES

In this section, we review some of the terminology used in [1]; for additional information we refer to [1] and to [2]. If G is an outerplanar graph we will always assume that G is drawn in the plane so that all the vertices border the exterior of G. If u and w are adjacent vertices of G we let $[u, w]$ denote the arc from u to w. A connected graph G is called a chain if every block contains at most two cutpoints of G and every cutpoint lies on only two blocks. The length of a chain is the number of blocks which it contains.

If G is any graph and u a vertex of G then $G \sim \{u\}$ denotes the graph obtained from G by deleting u and all arcs incident with u. Such a subgraph is called a point deletion or a (-1)-subgraph of G. If v is a vertex of G distinct from u then $G \sim \{u\}$ with the vertex v labeled is called a $(-1, v)$-subgraph of G.

Let G be a 2-connected outerplanar graph. Then the vertices of G lie on a unique spanning cycle and if we choose an initial vertex v of G and a direction of traversing the spanning cycle starting at v we say that we have selected an orientation of G. G with a selected orientation is said to be oriented. If G has an orientation with starting vertex v then G can be drawn in the plane so that the vertices of G are uniformly spaced around a circle with v at the "top." It is this "circular representation" which we have in mind when we use terms such as diameter through v, interior arc above, or below. Also, the neighbors of v mean the neighbors with respect to the cyclic ordering.

If G is a 2-connected outerplanar graph then $T(G)$ denotes the dual of G with that vertex of the dual deleted which corresponds to the exterior region of G. $T(G)$ is a tree. If G is any outerplanar graph then $T(G)$ means the disjoint union of the trees $T(H)$ for all 2-components H of G.

Again, let G be a 2-connected outerplanar graph. The one or two regions of G corresponding to the center of $T(G)$ is called the centrum of G. The 2-connected subgraphs of G corresponding to the branches of $T(G)$ at the center are called the arms of G at the centrum. A vertex of valence two in G is called extremal if the interior region of G on which it lies represents an extremal vertex of $T(G)$.

Finally, if G is any graph and if u and w are vertices of G which fail to be similar only because of a defect in the adjacency of u and w then $G \sim \{u\}$ and $G \sim \{w\}$ are clearly isomorphic; we say that vertices such as u and w are essentially similar.

3. Near Symmetry about a Vertex

Let G and H be v-labeled outerplanar blocks, $(v, v_1, \ldots, v_n)$ an orientation of G, $(v, w_1, \ldots, w_m)$ an orientation of H. By the v-concatenation (G, H) of G with H we mean the graph obtained from the disjoint union of G with H by identifying the vertices v of H and G, the vertex v_1, with w_m, and the arc $[v, v_1]$ with the arc $[v, w_m]$. The v-concatenation $(G_1, G_2, ..., G_t)$ of t blocks is similarly defined.

If H_1 and H_2 are subgraphs of G which are blocks it is clear what is meant by saying that an isomorphism between H_1 and H_2 preserves or reverses orientation.

Let Q_1 be the v-labeled quadrilateral with orientation (v, q_1, q_2, q_3) and Q_2 the same graph but with v and q_2 joined by an arc. A vertex v' of G is called a Q_k-point ($k = 1$ or 2) if G is a v-concatenation of induced subgraphs of the form (A, Q_k, B) and v' corresponds to the vertex q_2. If v' is a Q_1-point or a Q_2-point it is called a Q-point.

G is said to be nearly symmetric about v if G is not symmetric about v but there exist distinct vertices v' and v'' different from v such that $G \sim \{v'\}$ and $G \sim \{G''\}$ are v-isomorphic. Points such as v' and v'' are then called symmetry points with respect to v.

LEMMA 1. *Let G be a v-labeled outerplanar block and suppose that G has two $(-1, v)$-subgraphs $G \sim \{u\}$ and $G \sim \{w\}$ which are v-isomorphic. Then either G is symmetric about v or else u and w are Q-points of G.*

Proof. Induction on the number of vertices of G, which we assume is at least seven. Let $G = (v, v_1, ..., v_n)$ be an orientation of G and let $u = v_k$ and $w = v_{n-k+1}$.

Case 1. v is not a cutpoint of $G \sim \{u\}$.

(a) $G \sim \{u\}$ is a block. If the v-isomorphism of $G \sim \{u\}$ with $G \sim \{w\}$ reverses orientation then G is obviously symmetric about v and so we assume that orientation is preserved. If $k = 1$ then v is adjacent to every vertex, a symmetric result. If $k \geqslant 2$ then v_{k-1} is adjacent to v_{k+1}, and hence to v_{k+2}, to v_{k+3}, etc. Similarly, v_{n-k+2} is adjacent to v_{n-k}, and hence to v_{n-k-1}, v_{n-k-2}, etc. But this contradicts the outerplanarity of G.

(b) Suppose $k = 1$. Then v is not adjacent to v_2 or v_{n-1} (otherwise we are in subcase (a)). If v is adjacent to any v_t ($3 \leqslant t \leqslant n - 2$) then we can recover the ordering of at least half the vertices of G on both $G \sim \{u\}$ and $G \sim \{w\}$, and thus G must be symmetric. Hence we may assume that v is a vertex of valence two. Notice that the only arcs which cross the diameter through v are horizontal ones. If the only such interior arc is $[v_1, v_n]$, then G is symmetric. If there are other such arcs, let the one with lowest $m \geqslant 2$ be $[v_m, v_{n-m+1}]$. If $m > 2$ then G is symmetric. If $m = 2$ then if the isomorphism between $G \sim \{u\}$ and $G \sim \{w\}$ reverses orientation on the subgraphs corresponding to the portion of G bounded above by $[v_2, v_{n-1}]$ it must follow that G is symmetric; if orientation is preserved then that portion of G has a symmetry which corresponds to rotation by one vertex so again we get a symmetric outcome.

(c) There exists an interior arc in G separating v from u and the closest such arc to u crosses the diameter through v. Let this arc be $[v_m, v_h]$ $(1 \leqslant m < k)$. It is then necessary that $h = n - m + 1$ and that the subgraph H of G bounded below by $[v_m, v_h]$ be symmetric about v. Let $\bar{G}$ be the graph obtained from G by identifying all the vertices of H and labeling the result w. If $\bar{G}$ is symmetric about w then, since G is outerplanar and since v_m and v_h have the same valence on G, it is easy to see that G is symmetric about v. If $k = m + 1$ then $\bar{G}$ is symmetric about v by the inductive hypothesis; if $k > m + 1$ then G is symmetric about w for the same reason unless both v_{k-1} and v_{k+1} are adjacent to v_m, which we now assume. The v-isomorphism between $G \sim \{u\}$ and $G \sim \{w\}$ then gives rise to an isomorphism of induced subgraphs of G

$$(v_m, v_{m+1}, \ldots, v_{n-k}, v_{n-m+1}) \to (v_m, v_{k+1}, v_{k+2}, \ldots, v_{n-m+1})$$

and we may as usual assume that this isomorphism is orientation preserving. It is then easy to see that v_m is adjacent to all v_t for $t \leqslant h$ of the form $k + 1 + s(k - m)$ and v_h is adjacent to all v_t for $t \geqslant m$ of the form $n - k - s(k - m)$. But again this contradicts the fact that G is outerplanar.

(d) There exists an interior arc in G separating v from u and the arc of this type lying closest to u does not cross the diameter through v. Again, we may assume that this arc does not connect v_{k-1} with v_{k+1}. We can recover the ordering of at least half the vertices of G on $G \sim \{u\}$ and $G \sim \{w\}$ and so get a symmetric result.

Case 2. v is a cutpoint of $G \sim \{u\}$.

(a) The shorter branch at v on $G \sim \{u\}$ consists of a single block say H_1. The longer branch then contains a v-isomorphic block H_2. We identify the vertices of $H_1 \cup H_2$ and use the inductive hypothesis.

(b) The shorter branch at v on $G \sim \{u\}$ contains a nonterminal block H_1 containing v. Let v_m be the vertex of H_1 furthest from v in the ordering of G. Again, there exists an H_2 as in (a). Notice that $k > m + 1$. If v_{k-1} and v_{k+1} are not both adjacent to vertices in the set $\{v, v_m\}$ we can proceed by identifying the vertices in $H_1 \cup H_2$ and using the inductive hypothesis. If both v_{k-1} and v_{k+1} are adjacent to v_m the argument is identical with that of Case 1(c).

There remains the case where v_{k-1} is adjacent to v_m and v_{k+1} is adjacent to v. $G \sim \{u\}$ then consists of a larger block H_1 and a smaller branch L_1 intersecting in v as a cutpoint, and $G \sim \{w\}$ is similarly constructed from an H_2 and an L_2. We may assume that the isomorphism between $G \sim \{u\}$ and $G \sim \{w\}$ is orientation preserving between H_1 and H_2.

By utilizing all the arcs in G incident with v we may represent G as a maximal v-concatenation of induced subgraphs $(G_1, G_2, ..., G_a)$. Using the order preserving v-isomorphism between H_1 and H_2 one sees that actually G has the form $(G_1, ..., G_b, G_1, ..., G_b, G_1, ..., G_b, G_1, ..., G_c)$ for some c such that $1 \leqslant c \leqslant b$ and that H_1 has the same form but with one less copy of $(G_1, ..., G_b)$, and where $(G_1, ..., G_b)$ is the induced subgraph of G with vertices $v, v_1, ..., v_{k+1}$. Now consider the v-isomorphism between L_1 and L_2. If v_{k-1} is not adjacent to v then both $(G_1, ..., G_c)$ and $(G_{c+1}, ..., G_b)$ are symmetric about v whence so is G; if v_{k-1} is adjacent to v there is the additional possibility that u is a Q-point.

THEOREM 1. *Let G be a v-labeled oriented outerplanar block which is nearly symmetric about v. Then for $k = 1$ or 2, but not both, G is a v-concatenation of the form $(A, Q_k, A, Q_k, ..., Q_k, A)$ (at least three A's) for an appropriate subgraph A of G. If one expresses G as a maximal concatenation of this form then the symmetry points are just the Q_k-points.*

Proof. Let u be a symmetry point of G. By Lemma 1, u is a Q_k-point for $k = 1$ or 2. Using this k, express G as a maximal v-concatenation $(B_1, Q_k, B_2, Q_k, ..., Q_k, B_m)$. If u occurs in this representation before its companion symmetry point and if it occurs as the q_2 in the hth copy of Q_k then we have v-isomorphisms of induced subgraphs of G as follows:

$$(B_1, Q_k, ..., Q_k, B_h) \approx (B_{m-h+1}, Q_k, ..., Q_k, B_m)$$
$$(B_{h+1}, Q_k, ..., Q_k, B_m) \approx (B_1, Q_k, ..., Q_k, B_{m-h}).$$

Both these isomorphisms must preserve orientation in G, otherwise G would be symmetric about v. It follows that if d is the greatest common divisor of m and h and if A is the v-concatenation $(B_1, Q_k, ..., Q_k, B_d)$ then G is a v-concatenation $(A, Q_k, ..., Q_k, A)$. The other statements of the theorem are now immediate.

COROLLARY 1. *Let G be an outerplanar block. Then there exists at most one vertex v such that G is nearly symmetric about v.*

COROLLARY 2. *Let G be an outerplanar block, x, y, u, and w distinct vertices of G with x and y consecutive and suppose that there exists an isomorphism $\sigma: G \sim \{u\} \to G \sim \{w\}$ such that $\sigma(x) = y$ and $\sigma(y) = x$. Then G has a reflection interchanging x and y.*

Proof. Adjoin to G a new vertex v adjacent to only x and y and apply the theorem.

4. Schemes and Rosettes

Suppose that B is an oriented outerplanar block and let u and w be vertices of B with u the successor of w. We will then say that we have a based block with base $[u, w]$. If $n \geqslant 1$ and if B_j $(1 \leqslant j \leqslant n)$ is a based block with base $[u_j, w_j]$ then the graph obtained by identifying w_j with u_{j+1} $(1 \leqslant j \leqslant n-1)$ is called a based chain. Notice that a based chain has a natural orientation compatible with the orientations of its blocks. The above based chain is denoted $\langle B_1, B_2, ..., B_n \rangle$. If C is this based chain with $n \geqslant 2$ we obtain a based block, denoted $\hat{C}$, by connecting u_1 to w_n by an arc. If B is a based block having at least three vertices we obtain a based chain, denoted $\tilde{B}$, by deleting the base of B (except for its vertices) and using the orientation of B in the natural way. A based chain is said to be asymmetrical if it has no symmetry mapping base edges to base edges.

If G is an oriented outerplanar block having a single centrum Z and if c is a central vertex of G then G_c denotes the obvious based chain made up from the arms of G at Z, taken in the direction of orientation and beginning at c. Let $C = \langle B_1, ..., B_n \rangle$ and $C' = \langle B_1', ..., B_n' \rangle$ be based chains. Suppose that there exists an isomorphism $C \to C'$ which sends B_k to B_k' for each k $(1 \leqslant k \leqslant n)$. If in addition orientation is preserved between B_1 and B_1' we write $C \cdot\!\approx C'$; $C \approx\!\cdot\, C'$ and $C \cdot\!\approx\!\cdot\, C'$ are interpreted similarly. If E and F are based chains then $\langle E, F \rangle$ and $k\langle E \rangle$ (k a non-negative integer) have the obvious meanings. Note that, for convenience, we are allowing the empty set to be a based chain. If C is a based chain having at least three vertices and if B is the first block of C and $[u, w]$ the base of B then we denote by $^*C$ the based chain obtained from C by deleting u; C^* is similarly defined. A based chain $C = \langle B_1, ..., B_n \rangle$ is called a k-chain $(0 \leqslant k \leqslant n)$ if $\langle B_1, ..., B_k \rangle \cdot\!\approx\!\cdot \langle B_{n-k+1}, ..., B_n \rangle$ and if C is both a k-chain and an l-chain then C is called a (k, l)-chain. The following results are clear.

Lemma 2. *Let C be a (k, l)-chain of length $k + l$, let d be the greatest common divisor of k and l, and let D be the initial subchain of length d in C. Then $C \cdot\!\approx\!\cdot [(k + l)/k]\langle D \rangle$.*

Lemma 3. *Let C be a (k, l)-chain of length n with $k, l \geqslant \frac{1}{2}n$, $k \neq l$, and let d be the greatest common divisor of $n - k$ and $n - l$. Then $C \cdot\!\approx\!\cdot \langle h\langle D \rangle, E \rangle$, where D is the initial subchain of length d in C and E is an initial subchain in D.*

Let $X = \langle B_1, ..., B_n \rangle$ be a (k, l)-chain of length n, and let B be a based block. Suppose that $t = n - k - l - 1 > 0$, that $B \cdot \approx \cdot B_{k+1} \cdot \approx \cdot B_{l+t+1}$, and that if $C' = \langle B_{k+2}, \ldots, B_{k+t+1} \rangle$ and $C'' = \langle B_{l+1}, \ldots, B_{l+t} \rangle$ then B is not isomorphic with any block in C' or C''. We will then say that X is a scheme with parameters (n, k, l) and data (B, C', C''). We will now classify all schemes. A scheme contains subchains $\langle C'', B \rangle$ and $\langle B, C' \rangle$ which may overlap by at least one block or may be separated by $r \geqslant 0$ blocks. In the first case let us say that the scheme is reduced. Suppose that X is not reduced and let $M = \max\{l, k\}$ and $m = \min\{l, k\}$. Then $X' = \langle B_1, \ldots, B_M \rangle$ is a scheme which parameters (M, r, m) or (M, m, r) and data (B, C', C''). X' is called the contraction of X. Notice that X' can be subjected to an obvious expansion operation in at least two ways (see below), yielding X and at least one other scheme. Since $M < n$ repeated contraction of X must yield a reduced scheme. However, reduced schemes can be classified directly, and this classification yields the following result.

PROPOSITION 1. *Let X be a scheme with data (B, C', C''). Then X is obtained by repeated expansions from one of the following based chains Y:*

(1) *$C' = \langle D, E \rangle$, $C'' = \langle E, D \rangle$ (where D but not E may be empty), and $Y = \langle S, B, S \rangle$ with $S = \langle D, d\langle C'' \rangle \rangle$ $(d \geqslant 1)$.*

(2) *With C' and C'' as above, $Y = \langle S, E, S \rangle$ with $S = \langle D, d\langle B, D \rangle \rangle$ $(d \geqslant 1)$.*

COROLLARY. *Suppose that X is a scheme with parameters (n, k, l) and (n, k', l'), both with data (B, C', C''). Then necessarily (k, l) and (k', l') are equal. Hence, a scheme with data (B, C', C'') has only two expansions with the same data.*

Proof. Examination of the reduced schemes given in Proposition 1 shows that X cannot be reduced with respect to either of the given parameter sets. Hence either k or l is $\geqslant \frac{1}{2}n$ and either k' or l' is $\geqslant \frac{1}{2}n$. We will assume that $l > l' \geqslant \frac{1}{2}n$ and get a contradiction, the proofs in other cases being similar. If d is the greatest common divisor of $k + t + 1$, $k' + t + 1$, and l then by Lemmas 3 and 2 X consists of a number of repeats of a subchain of length d of the form $\langle Y, B, C' \rangle$. This gives a contradiction.

An oriented outerplanar block G is called a rosette of the first kind if:

(a) G has a single centrum and has central vertices u and w such that $G \sim \{u\}$ and $G \sim \{w\}$ are isomorphic under an isomorphism which preserves orientation on the nonterminal blocks of $G \sim \{u\}$.

(b) There exists an asymmetrical based block B and based chains C_1 and C_2 of the same length >1 such that $^*C_1 \approx\cdot\, ^*B$ and $C_2^* \cdot\!\approx B^*$, and there exists a based chain X such that:

$$G_u \cdot\!\approx\!\cdot \langle C_1, X, C_2, k\langle B, X, C_2 \rangle, L_1, B \rangle \qquad (k \geqslant 0),$$

where w is the terminal vertex of the first C_2 occurring and where $\langle L_1, B\rangle$ is an initial subchain of $\langle B, X, C_2\rangle$; $G_w \cdot\!\approx\!\cdot \langle B, L_2, k\langle C_1, X, B\rangle, C_1, X, C_2\rangle$ where $\langle B, L_2\rangle$ is a terminal subchain of $\langle C_1, X, B\rangle$.

The simultaneous existence of decompositions of these types clearly puts strong restrictions on the possible forms of C_1, C_2, and X. An analysis of the required compatibility conditions yields the following result, which effectively classifies all rosettes of the first kind.

PROPOSITION 2. *Let G be a rosette of the first kind and use the notations above. Then one of the following holds:*

(1) $C_1 = C_2 = \tilde{B}$; $X = L_1 = d\langle B\rangle$ $(d \geqslant 0)$.

(2) *There exist based chains F and G such that:* $C_1 = \langle F, G\rangle$; $C_2 = \langle G, F\rangle$; $X = \langle F, d\langle G, F\rangle\rangle$ $(d \geqslant 0)$; *L_1 is empty.*

(3) *With C_1 and C_2 as in* (2): $X = \langle F, d\langle B, F\rangle\rangle$ $(d \geqslant 1)$; $L_1 = d\langle B, F\rangle$.

(4) *X is a (k, l)-scheme with data (B, C_1, C_2) and $L_1 = \langle B, A\rangle$, where A is the initial subchain of length k in X.*

Again, let B be an asymmetric based block and now let E and F be based chains such that F is nonempty and $*B \approx\!\cdot *\langle E, F\rangle$ and $B^* \cdot\!\approx \langle F, E\rangle^*$. Let G be an outerplanar block with single centrum such that $G_u \cdot\!\approx\!\cdot \langle E, F, k\langle E, B\rangle\rangle$ $(k \geqslant 1)$ for some central vertex u. If w is the terminal vertex of the second copy of E occuring then $G \sim \{u\}$ and $G \sim \{w\}$ are isomorphic under an orientation preserving isomorphism. Such a G is called a rosette of the second kind.

A rosette of the first or of the second kind with or without an added arc connecting u and w (provided that this are does not give a symmetric graph) will be simply called a rosette. Notice that if adding an arc between u and w in a rosette of the first or second kind gives a symmetric graph then u and w are similar points of this new graph. It is a consequence of our definitions that a rosette never has a symmetry. The vertices u and w introduced above are called symmetry points. It will follow from Lemma 4 proved below that if a rosette has two isomorphic one point deleted subgraphs then the deleted vertices must be central. Using this fact it is easy to check that the only two isomorphic one point deleted subgraphs of a rosette are $G \sim \{u\}$ and $G \sim \{w\}$.

5. THE MAIN THEOREM

LEMMA 4. *Let G be an oriented outerplanar block and let u and w be distinct noncentral vertices of G such that there exists an isomorphism*

σ: $G \sim \{u\} \to G \sim \{w\}$. Then either u and w are similar points or else G is nearly symmetric about some vertex.

Proof. Let H_1 be the (unique) 2-component of $G \sim \{u\}$ such that $T(H_1)$ has maximal length and H_2 similarly on $G \sim \{w\}$. Let Z denote the centrum of G.

Case 1. Length $T(H_1)$ = length $T(G)$. Then σ induces a rotation or reflection on Z. If σ induces a reflection on Z, then σ is the restriction to $G \sim \{u\}$ of a reflection of G. If σ induces a rotation on Z, then this rotation must be nontrivial, otherwise u and w lie on the same arm of G at Z and this yields a contradiction.

Suppose that Z has m vertices and let the action of σ on Z be rotation by $k > 0$ vertices. If d is the greatest common divisor of m and k, then G has a rotational symmetry which corresponds to rotation by d vertices on Z, and σ is the restriction to $G \sim \{u\}$ of a power of this symmetry.

Case 2. length $T(H_1) <$ length $T(G)$. Then $T(G)$ has exactly two branches of maximal length, B_1 and B_2, at its center, with corresponding arms A_1 and A_2.

Subcase (*a*). H_1 is a nonterminal block on $G \sim \{u\}$. If e_1 is the base edge of H_1 and e_2 similarly for H_2 then $\sigma(e_1) = e_2$ and from this it follows that σ is the restriction to $G \sim \{u\}$ of a reflection or a rotation of order two.

Subcase (*b*). $G \sim \{u\}$ has at least two blocks with H_1 a terminal block. If both u and w lie on say A_1, a count of regions on H_1 and H_2 shows that no interior arc of G separates u from w, i.e., that u and w lie on the same region of G, and so $H_1 = H_2$. If σ is orientation reversing on H_1, u and w are similar under a reflection of G. If σ is orientation preserving on H_1 it is clear that σ consists of rotation by one vertex there, so that H_1 is a polygon and G itself consists of only two regions, a contradiction. Hence we may assume that u lies on A_1 and w on A_2.

We will now suppose that σ is orientation preserving on H_1, the other case being similar. Let c_u be the cutpoint of $G \sim \{u\}$ on H_1, and c_w similarly defined on H_2, and assume that $c_u \neq c_w$ since otherwise G is nearly symmetric about c_u. If these four points lie in the order (c_u, u, c_w, w) in the orientation of G then σ can be extended to a rotation of order two on G. If these four points lie say in the order (c_u, u, w, c_w) let u' be the successor of u and w' the predecessor of w on G. By comparing maximal paths in $T(H_1)$ beginning at c_u (i.e., beginning at any region having c_u as a vertex) with maximal paths in $T(H_2)$ beginning at c_w one shows that $\sigma([w', c_w]) = [c_u, u']$ and this contradicts the fact that σ is orientation preserving.

Subcase (*c*). $G \sim \{u\}$ is a block. Again we may assume that u lies on A_1 and w lies on A_2 and in that case these are the unique extremal vertices on their respective arms.

Suppose first that σ is order preserving. The cases where G has a single or double centrum being similar we will only examine the case of a single centrum. Then $G \sim \{u\}$ has a double centrum $\{Z, P_1\}$ and $G \sim \{w\}$ has a double centrum $\{Z, P_2\}$. If $\sigma(Z) = Z$ one sees easily that σ is the restriction to $G \sim \{u\}$ of a rotation of order two of G. In the contrary case we must have $\sigma(Z) = P_2$ and $\sigma(P_2) = Z$. From this it follows that the subgraph of G corresponding to the maximal path in $T(G)$ is a fan. Since the effect of σ on this fan is translation by one triangle one sees finally that this fan is all of G.

Lastly, consider the case where σ is order reversing. To begin with, let $(v_1, ..., v_n)$ be the given orientation of G, suppose that $u = v_1$, $w = v_k$, and examine the following configurations separately: n even, $k = \frac{1}{2}(n + 2)$, and $\sigma(v_2) = v_1$ or v_n; n odd, $k = \frac{1}{2}(n + 3)$, and $\sigma(v_2) = v_n$; n odd, $k = \frac{1}{2}(n + 1)$, and $\sigma(v_2) = v_n$. In each of these, $T(G)$ is a path, each region of G is a triangle, and u and w are similar points. In all other configurations there exists a vertex v such that $\sigma(v) = v$ or else there exist a pair of consecutive vertices x and y such that $\sigma(x) = y$ and $\sigma(y) = x$.

THEOREM 2. *Let G be an outerplanar block, u and w distinct vertices of G such that $G \sim \{u\}$ and $G \sim \{w\}$ are isomorphic. Then one of the following cases holds*:

(1) *u and w are similar or essentially similar*;

(2) *G is nearly symmetric about some vertex v and u and w are symmetry points with respect to v*;

(3) *G is a rosette with u and w symmetry points.*

Proof. First, select an orientation for G. By Lemma 4 we may suppose that u and w are central vertices of G and hence that $G \sim \{u\}$ consists of at least three blocks.

We will first consider the case where σ is orientation reversing on the nonterminal blocks of $G \sim \{u\}$ and, the case of a single centrum being the more interesting, we will examine that. We may suppose that the terminal blocks of $G \sim \{u\}$ contain more that half the vertices of G, otherwise σ would leave fixed a vertex of G or interchange a pair of consecutive vertices.

If both arms of G at Z containing u are trivial then it is easy to see that (provided that Z has >5 vertices) σ leaves fixed a vertex of Z or interchanges a pair of consecutive vertices of Z. If at least one of the arms of

G at Z containing u is nontrivial then we can apply our size assumption to conclude that u and w must be consecutive vertices of Z or separated by at most one vertex of Z, this vertex having valence two. This latter case yields a symmetric outcome and so we may assume that say w immediately follows u on Z.

Denote by A_1 and A_2 the arms at Z containing u and by A_2 and A_3 the arms containing w. We may clearly assume A_2 to be nontrivial. Then the initial block B_1 of $G \sim \{u\}$ and the final block B_2 of $G \sim \{w\}$ are both contained in A_2 and, if $\bar{u}$ is the successor of u and $\bar{w}$ the predecessor of w on G, our size assumption implies that $\bar{u}$ and $\bar{w}$ are adjacent, and that $B_1 = B_2$ is the subgraph of G obtained from A_2 by deleting u and w. It is now easy to see that $\sigma(w) = u$ whence G has a reflection interchanging u and w.

Next we assume that σ is orientation preserving on the nonterminal blocks of $G \sim \{u\}$. In addition, we will suppose for the moment that G has a single centrum Z.

Let $\{e_1, e_2, ..., e_m\}$ denote the edges of Z written in an order compatible with the orientation of G and assume that u is shared by e_m and e_1 and w by e_l and e_{l+1}. For each i let B_i be the arm of G at e_i or, if G has a trivial arm at e_i, let B_i be just e_i. Then $G \sim \{u\}$ and $G \sim \{w\}$ are the following based chains:

$$\langle {}^*B_1, B_2, ..., B_m{}^* \rangle \qquad \langle {}^*B_{l+1}, B_{l+2}, ..., B_m, B_1, ..., B_l{}^* \rangle$$

We may assume w.l.o.g. that

$${}^*B_{l+1} = \sigma(\langle {}^*B_1, B_2, ..., B_a \rangle) \qquad (a \geqslant 1)$$

and then necessarily

$$\langle B_{l-a+1}, ..., B_l{}^* \rangle = \sigma(B_m{}^*)$$

and $a \leqslant l$ and $\sigma(B_k) = B_{k+l-a+1} (a + 1 \leqslant k \leqslant m - 1)$, where we make the convention that all subscripts are to be replaced by their least positive residues modulo m.

By applying powers of σ to B_{l+1} we see that B_{l+1} is $\cdot\approx\cdot$ with some block in the list $\{B_1, ..., B_a, B_m\rangle$ and similarly that B_m is $\cdot\approx\cdot$ with some block in the list $\{B_{l+1}, B_{l-a+1}, ..., B_l\}$. If $a = 1$, one possibility is that $B_1 \cdot\approx\cdot B_{l+1}$ and $B_m \cdot\approx\cdot B_l$ and this yields a rotationally symmetric outcome; the other possibility when $a = 1$, and by size considerations the only possibility when $a > 1$, is $B_m \cdot\approx\cdot B_{l+1}$, which we now assume. Let B denote B_{l+1}, let $C_1 = \langle B_1, ..., B_a \rangle$ and $C_2 = \langle B_{l-a+1}, ..., B_l \rangle$, and assume for the moment that $a < l - a + 1$. If $X = \langle B_{a+1}, ..., B_{l-a} \rangle$ then $G \sim \{u\}$ has the form $\langle {}^*C_1, X, C_2, k\langle B, X, C_2 \rangle, L_1, B^* \rangle$, where $k \geqslant 0$ and $\langle L_1, B \rangle$ is an initial subchain in $\langle B, X, C_2 \rangle$. Likewise, $G \sim \{w\}$ has the form $\langle {}^*B, L_2, k\langle C_1, X, B \rangle, C_1, X, C_2{}^* \rangle$, where $\langle B, L_2 \rangle$ is a terminal subchain of $\langle C_1, X, B \rangle$. If the based block B is symmetrical then u and w are similar; otherwise, G is a rosette of the first kind.

If $a \geqslant l - a + 1$ the proof is similar except that in that case "X" is absent, C_1 and C_2 overlap by at least a block and we arrive at a rosette of the second kind.

This completes the proof except in the configuration where σ is order preserving on nonterminal blocks and G has a double centrum. We then let P_1 and P_2 be the polygons constituting Z and let $[v_1, v_2]$ be their common edge. If $u = v_1$, say, then $w = v_2$ and we may apply our previous arguments to the graph obtained from G by deleting the arc connecting v_1 and v_2 (except for the vertices). Hence we assume that neither u nor w is v_1 or v_2. If u and w lie on say P_1 then necessarily u is adjacent to say v_1 and w to v_2 and there are no arms on $[v_1, u]$ and $[v_2, w]$. But then we find that σ is the restriction to $G \sim \{u\}$ of a reflection and so G does not conform to the present requirements. If say u lies on P_1 and w on P_2 then u and w are similar under a rotation of order two of G.

6. An Application

LEMMA 5. *Let G be an outerplanar block. Let u and w be essentially similar points of G. Then $G \sim \{u\}$ and $G \sim \{w\}$ are the only isomorphic (-1)-subgraphs of G.*

Proof. If u and w are not adjacent in G then G is a rosette and so the result is true. Hence we may assume that u and w are adjacent in G and also that the graph obtained from G by deleting the arc from u to w has a rotation but no reflection sending u to w and has a single centrum. The present result then follows easily from Lemma 4.

LEMMA 6. *Let G be an outerplanar block which is known to have only a reflection with one or two fixed points as a symmetry. Then G can be reconstructed from the isomorphism classes of its (-1)-subgraphs.*

Proof. Let Z denote the centrum and r_1 the reflection of G. Except in certain easily handled special cases there exists a noncentral vertex v of G lying on an arm Q of G at Z and having the following properties: $r_1(Q) \neq Q$; Z can be identified on $G \sim \{v\}$; the location of Q on $G \sim \{v\}$ can be identified. Of course, Q is not intact on $G \sim \{v\}$ but all other arms at Z are intact.

For ease of geometrical expression let us imagine G drawn so that the boundary of Z is a regular polygon. If we can identify the axis of r_1 on $G \sim \{v\}$ this will of course prove unique reconstructability. r_1 has the property that if we delete from $G \sim \{v\}$ the residue of Q and all of its symmetrically situated counterpart Q' then r_1 on the remaining subgraph is a reflection. Suppose now that this description does not specify r_1 uniquely, i.e., suppose that there exists another axis not through Q so that if we delete Q and its symmetric counterpart Q'' with respect to this axis then the reflection r_2 in this axis is a symmetry of the remaining subgraph.

If A is any arm of G at Z (considered as a based block relative to some orientation of G) and if $A \neq Q, Q''$ then $r_1(r_2(A))$ is also an arm, which we denote by $R(A)$, and $R(A) \cdot\!\approx\!\cdot A$. If $R(A)$ is defined, then $R(A)$ is obtained from A by counting k arms in the direction of orientation of G, where k is independent of A. It is now easy to verify that R represents a rotation of G, and so we have reached a contradiction, unless R is the identity so that r_1 and r_2 coincide.

THEOREM 3. *Let G be a 2-connected outerplanar graph. Then G can be reconstructed from the isomorphism classes of its (-1)-subgraphs.*

Proof. If G has n vertices then there are exactly n/k $(k \geqslant 2)$ isomorphism classes if and only if G has a rotational symmetry or (if $k = 2$) a reflection without fixed points. We may then recover the entire collection of (-1)-subgraphs of G by taking each nonisomorphic one with multiplicity k. If n is odd (resp. even) and there are exactly $\frac{1}{2}(n + 1)$ (resp. $\frac{1}{2}n + 1$) isomorphism classes then G has a reflection with one (resp. two) fixed points and we can use Lemma 6.

In all other cases where we do not have a full complement of distinct (-1)-subgraphs G must either be symmetric about some v, or be a rosette, or else G has a pair of essentially similar points.

If G is nearly symmetric about a vertex v one can find a subgraph corresponding to deleting a Q-point and clearly such a graph can have a vertex of valence two or three adjoined to it in only one way in order to yield a graph which has a repeated subgraph.

Suppose now that G has a pair of essentially similar vertices u and w and make the same assumptions about G as in the proof of Lemma 6. G then has a double centrum say $\{P_1, P_2\}$ with $[u, w]$ the shared line. If P_1 and P_2 have equally many vertices then G has a symmetry, a case we are presently excluding; if P_1 and P_2 are of different sizes, reconstruction can be accomplished from subgraphs corresponding to deleting central vertices other than u and w on P_1 and P_2.

In all remaining cases of multiplicity of subgraphs G must be a rosette. But, using the notation of our discussion of rosettes, G then has the property that all its arms of maximal length at Z are $\cdot\approx\cdot B$ (first kind) or $\cdot\approx\cdot D$ (second kind). Hence we may reconstruct from $G \sim \{v\}$, v an extremal vertex on B or D.

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